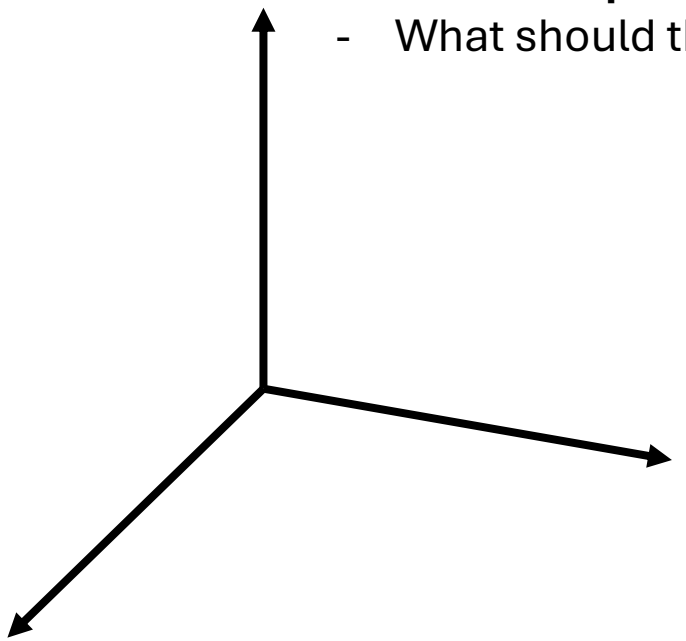
Machine Programming

Lecture 3 – Type Systems and Top-Down Enumerative Synthesis

Ziyang Li

Dimensions in Program Synthesis

Behavioral Specification
- What should the program do?



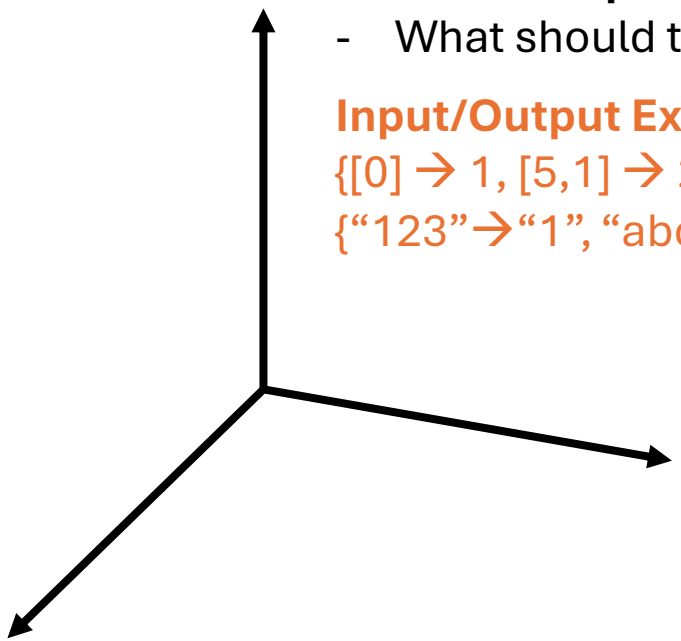
Synthesis Strategy
- How do we find such a program?

Structural Specification
- What is the space of the programs?

Last Week

Behavioral Specification
- What should the program do?

Input/Output Examples
{[0] → 1, [5,1] → 2}
{“123”→“1”, “abc”→“a”}



Synthesis Strategy
- How do we find such a program?

Enumeration
- Enumerating all programs with a grammar
- Bottom-up vs top-down

Structural Specification
- What is the space of the programs?

Context-Free / Regular Tree Grammar
Expr e ::= c | e + e | e * e

Last Week

Behavioral Specification
- What should the program do?

Input/Output Examples
{[0] → 1, [5,1] → 2}
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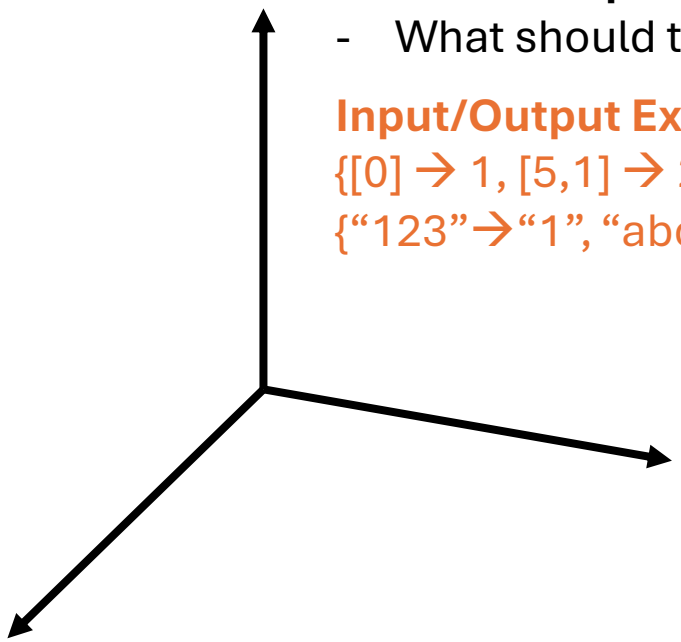


Synthesis Strategy
- How do we find such a program?

Enumeration
- Enumerating all programs with a grammar
- Bottom-up vs top-down

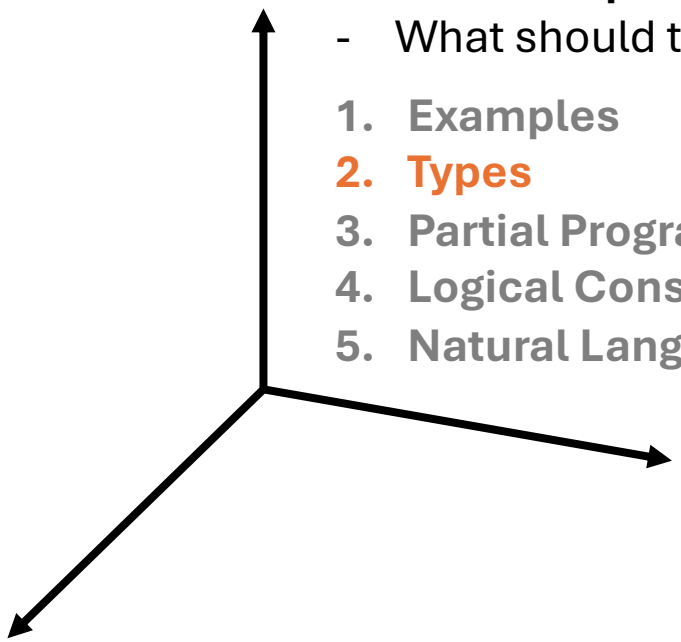
Structural Specification
- What is the space of the programs?

Context-Free / Regular Tree Grammar
Expr e ::= c | e + e | e * e

Today

Behavioral Specification
- What should the program do?

1. **Examples**
2. **Types**
3. **Partial Programs**
4. **Logical Constraints**
5. **Natural Language**



Synthesis Strategy
- How do we find such a program?

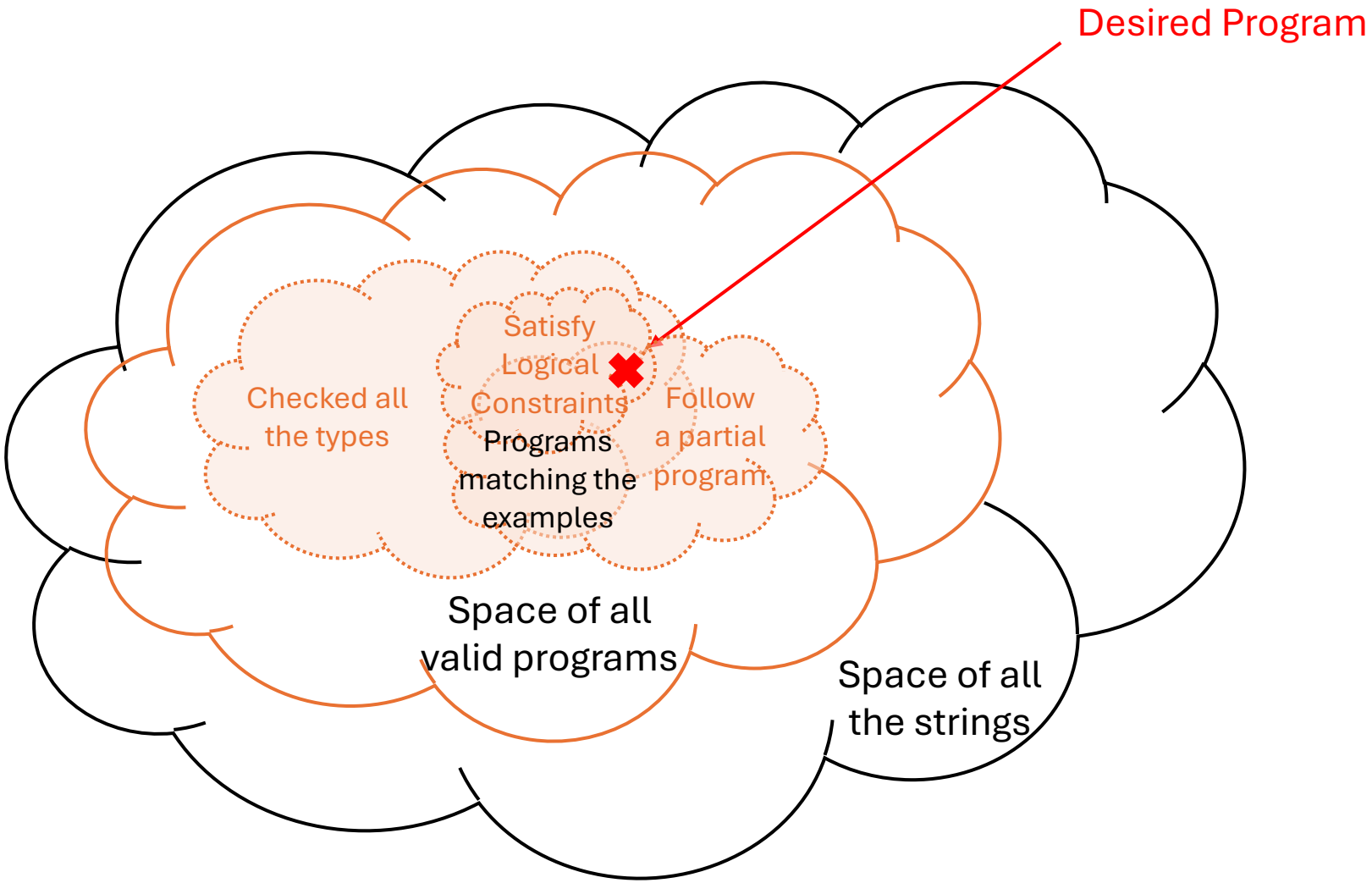
Enumeration
- Enumerating all programs with a grammar
- Bottom-up vs top-down

Structural Specification
- What is the space of the programs?

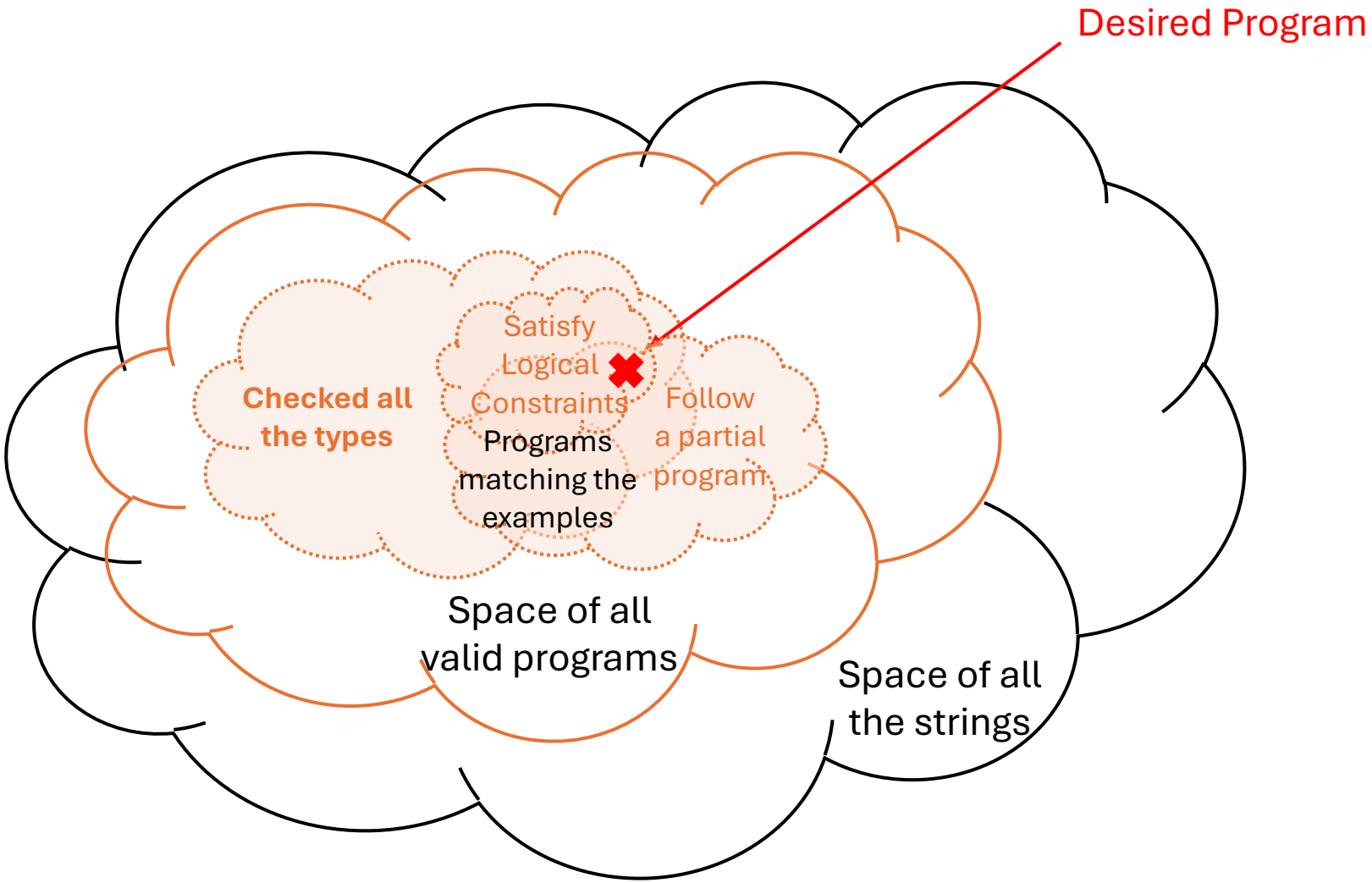
Context-Free / Regular Tree Grammar

Expr e ::= c | e + e | e * e

High Level Picture



High Level Picture



Specification: Types

- Fundamentals
  - Example: A Domain Specific Language (DSL) on List Manipulation
  - Type System
  - Type Checking
  - Pruning Top-Down Search with Types
- Advanced
  - Polymorphic Types
  - Refinement Types

List Manipulation DSL

Synthesis Goal:

Input

```
[ [71, 75, 83],
  [90, 87, 95],
  [68, 77, 80] ]
```

Output

```
[ [75, 83],
  [90, 95],
  [77, 80] ]
```

List Manipulation DSL

Synthesis Goal:

Input

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[ [71, 75, 83],
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[ [75, 83],
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```

`dropmins`: for each inner list corresponding to a list of grades, drop the lowest grade

List Manipulation DSL

`dropmins`: for each inner list corresponding to a list of grades, drop the lowest grade

Input	Output

`[ [71, 75, 83],`
`  [90, 87, 95],`
`  [68, 77, 80] ]`
	`[ [75, 83],`
	`  [90, 95],`
	`  [77, 80] ]`

```
def dropmins(input: List[List[int]]) -> List[List[int]]:
  # implementation
```

List Manipulation DSL

`dropmins`: for each inner list corresponding to a list of grades, drop the lowest grade

Input

```
[ [71, 75, 83],
  [90, 87, 95],
  [68, 77, 80] ]
```

Output

```
[ [75, 83],
  [90, 95],
  [77, 80] ]
```

```
def dropmins(input: …) -> …:
  outputs = []
  for grades in input:
    dropmin = []
    min_num = min(grades)
    for grade in grades:
      if grade > min_num:
        dropmin.append(grade)
    outputs.append(dropmin)
  return outputs
```

List Manipulation DSL

`dropmins`: for each inner list corresponding to a list of grades, drop the lowest grade

Input

```
[ [71, 75, 83],
  [90, 87, 95],
  [68, 77, 80] ]
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def dropmins(input: …) -> …:
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  for grades in input:
    dropmin = []
    min_num = min(grades)
    for grade in grades:
      if grade > min_num:
        dropmin.append(grade)
    outputs.append(dropmin)
  return outputs
```

```
def dropmins(input: …) -> …:
  outputs = []
  for grades in input:
    dropmin = [g for g in grades if g > min(grades)]
    outputs.append(dropmin)
  return outputs
```

List Manipulation DSL

`dropmins`: for each inner list corresponding to a list of grades, drop the lowest grade

Input

```
[ [71, 75, 83],
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def dropmins(input: …) -> …:
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        dropmin.append(grade)
    outputs.append(dropmin)
  return outputs
```

```
def dropmins(input: …) -> …:
  outputs = []
  for grades in input:
    dropmin = [
      g for g in grades
      if g > min(grades)
    ]

    outputs.append(dropmin)
  return outputs
```

List Manipulation DSL

`dropmins`: for each inner list corresponding to a list of grades, drop the lowest grade

Input

```
[ [71, 75, 83],
  [90, 87, 95],
  [68, 77, 80] ]
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Output

```
[ [75, 83],
  [90, 95],
  [77, 80] ]
```

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def dropmins(input: …) -> …:
  outputs = []
  for grades in input:
    dropmin = []
    min_num = min(grades)
    for grade in grades:
      if grade > min_num:
        dropmin.append(grade)
    outputs.append(dropmin)
  return outputs
```

```
def dropmins(input: …) -> …:
  outputs = []
  for grades in input:
    dropmin = [
      g for g in grades
      if g > min(grades)
    ]
    outputs.append(dropmin)
  return outputs
```

```
def dropmins(input: …) -> …:
  outputs = [
    [
      g for g in grades
      if g > min(grades)
    ]
    for grades in input
  ]
  return outputs
```

List Manipulation DSL

`dropmins`: for each inner list corresponding to a list of grades, drop the lowest grade

Input

```
[ [71, 75, 83],
  [90, 87, 95],
  [68, 77, 80] ]
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Output

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[ [75, 83],
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def dropmins(input: …) -> …:
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      if grade > min_num:
        dropmin.append(grade)
    outputs.append(dropmin)
  return outputs
```

```
def dropmins(input: …) -> …:
  outputs = []
  for grades in input:
    dropmin = [
      g for g in grades
      if g > min(grades)
    ]
    outputs.append(dropmin)
  return outputs
```

```
def dropmins(input: …) -> …:
  return [
    [
      g for g in grades
      if g > min(grades)
    ]
    for grades in input
  ]
```

List Manipulation DSL

`dropmins`: for each inner list corresponding to a list of grades, drop the lowest grade

Input

```
[ [71, 75, 83],
  [90, 87, 95],
  [68, 77, 80] ]
```

Output

```
[ [75, 83],
  [90, 95],
  [77, 80] ]
```

```
def dropmins(input: …) -> …:
  return [
    [
      g for g in grades
      if g > min(grades)
    ]
    for grades in input
  ]
```

```
def dropmins(input: …) -> …:
  return [

    filter(grades, lambda g: g > min(grades))


    for grades in input
  ]
```

List Manipulation DSL

`dropmins`: for each inner list corresponding to a list of grades, drop the lowest grade

Input

```
[ [71, 75, 83],
  [90, 87, 95],
  [68, 77, 80] ]
```

Output

```
[ [75, 83],
  [90, 95],
  [77, 80] ]
```

```
def dropmins(input: …) -> …:
  return [
    filter(
      grades,
      lambda g: g > min(grades))
    for grades in input
  ]
```

```
def dropmins(input: …) -> …:
  return map(input, lambda grades:
    filter(grades, lambda g:
      g > min(grades)
    )
  )
```

List Manipulation DSL

`dropmins`: for each inner list corresponding to a list of grades, drop the lowest grade

Input

```
[ [71, 75, 83],
  [90, 87, 95],
  [68, 77, 80] ]
```

Output

```
[ [75, 83],
  [90, 95],
  [77, 80] ]
```

```
def dropmins(input: …) -> …:
  return map(input, lambda grades:
    filter(grades, lambda g:
      g > min(grades)
    )
  )
```

```
def dropmins(input: …) -> …:
  return map(input, lambda grades:
    filter(grades, lambda g:
      g > reduce(grades, 0, lambda acc, cur:
        acc if acc < cur else cur
      )
    )
  )
```

```
def dropmins(input: …) -> …:
  return map(input, lambda grades:
    filter(grades, lambda g:
      g > reduce(grades, 0, lambda acc, cur:
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      )
    )
  )
```

map

filter

reduce

```
def dropmins(input: …) -> …:
  return map(input, lambda grades:
    filter(grades, lambda g:
      g > reduce(grades, 0, lambda acc, cur:
        acc if acc < cur else cur
      )
    )
  )
```

```
map       : [U] -> (U -> V) -> [V]

filter    : [U] -> (U -> bool) -> [U]

reduce    : [U] -> V -> (V -> U -> V) -> V
```

```
map       : [U] -> (U -> V) -> [V]
```

Take a list of U, apply the mapping function U → V, obtaining a list of V.

```
filter    : [U] -> (U -> bool) -> [U]
```

Take a list of U, use the filtering function U → bool to keep only the Us that produces true. The result is again a list of U.

```
reduce    : [U] -> V -> (V -> U -> V) -> V
```

Take a list of U and an initial state V. From the left, continuously apply the transition function V → U → V to accumulate the state V. After traversing the entire list, return the final V.

Typing Practices

```
sum

min

len

sort
```

Typing Practices

```
sum          : [Int] -> Int

min          : [Int] -> Int

len          : [Int] -> Int

sort         : [Int] -> [Int]
```

Implementation Practices

```
sum          : [Int] -> Int

min          : [Int] -> Int

len          : [Int] -> Int

sort         : [Int] -> [Int]
```

Your Tools:

```
map        : [U] -> (U -> V) -> [V]
filter     : [U] -> (U -> bool) -> [U]
reduce     : [U] -> V -> (V -> U -> V) -> V
```

Implementation Practices

```
sum          : [Int] -> Int
  sum(list) = reduce(list, 0, lambda acc, n: acc + n)
min          : [Int] -> Int

len          : [Int] -> Int

sort         : [Int] -> [Int]
```

Your Tools:

```
map       : [U] -> (U -> V) -> [V]
filter    : [U] -> (U -> bool) -> [U]
reduce    : [U] -> V -> (V -> U -> V) -> V
```

Implementation Practices

```
sum          : [Int] -> Int
  sum(list) = reduce(list, 0, lambda acc, n: acc + n)
min          : [Int] -> Int
  min(list) = reduce(list, None, lambda acc, n: n if acc is None or n < acc else acc)
len          : [Int] -> Int

sort         : [Int] -> [Int]
```

Your Tools:

```
map       : [U] -> (U -> V) -> [V]
filter    : [U] -> (U -> bool) -> [U]
reduce    : [U] -> V -> (V -> U -> V) -> V
```

Implementation Practices

```
sum          : [Int] -> Int
  sum(list) = reduce(list, 0, lambda acc, n: acc + n)
min          : [Int] -> Int
  min(list) = reduce(list, None, lambda acc, n: n if acc is None or n < acc else n)
len          : [Int] -> Int
  len(list) = reduce(list, 0, lambda acc, n: acc + 1)
sort         : [Int] -> [Int]
```

Your Tools:

```
map       : [U] -> (U -> V) -> [V]
filter    : [U] -> (U -> bool) -> [U]
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Implementation Practices

```
sum          : [Int] -> Int
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len          : [Int] -> Int
  len(list) = reduce(list, 0, lambda acc, n: acc + 1)
sort         : [Int] -> [Int]
  ???
```

Your Tools:

```
map       : [U] -> (U -> V) -> [V]
filter    : [U] -> (U -> bool) -> [U]
reduce    : [U] -> V -> (V -> U -> V) -> V
```

```
dropmins(input) =
  map(input, lambda grades:
    filter(grades, lambda g:
      g > reduce(grades, 0, lambda acc, cur:
        acc if acc < cur else cur
      )
    )
  )
```

```
map       : [U] -> (U -> V) -> [V]

filter    : [U] -> (U -> bool) -> [U]

reduce    : [U] -> V -> (V -> U -> V) -> V
```

List Manipulation DSL Syntax

```
Expr ::= Var
       | lambda Var: Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
       | Expr if Expr else Expr
       | Expr BinaryOp Expr
       | true | false
       | 0 | 1 | 2 | …

Var ::= String

BinaryOp ::= > | < | == | + | - | …
```

Typing Checking

```
Expr ::= Var
       | lambda Var: Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
       | Expr if Expr else Expr
       | Expr BinaryOp Expr
       | true | false
       | 0 | 1 | 2 | …

Var ::= String

BinaryOp ::= > | < | == | + | - | …
```

```
true + map(3, list)

1 + (lambda x: x + 1)

map(input, lambda x: x + 1)
```

Typing Checking

```
Expr ::= Var
       | lambda Var: Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
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Var ::= String

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```
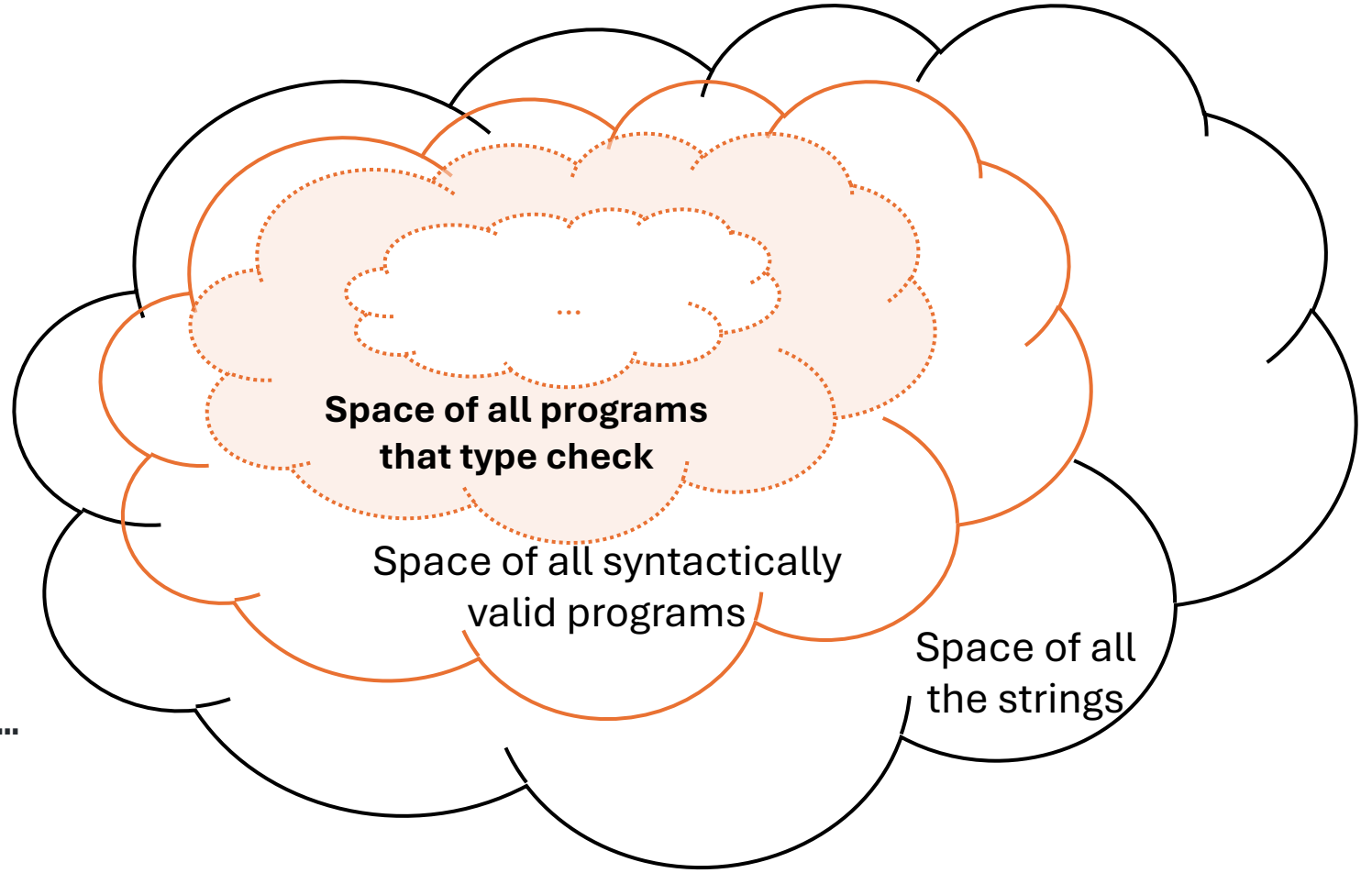



Types

```
Expr ::= Var
       | lambda Var: Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
       | Expr if Expr else Expr
       | Expr BinaryOp Expr
       | true | false
       | 0 | 1 | 2 | …

Var ::= String

BinaryOp ::= > | < | == | + | - | …
```

```
Type ::= Int
       | Bool
       | List[Type]
       | Type -> Type
```

Typing Rule

```
Expr ::= Var
       | lambda Var: Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
       | Expr if Expr else Expr
       | Expr BinaryOp Expr
       | true | false
       | 0 | 1 | 2 | …
```

```
Type ::= Int
       | Bool
       | List[Type]
       | Type -> Type
```

$$\frac{\text{<Premises>}}{\text{<Context> } \vdash \text{ <Expr> : <Type>}}$$

Typing Rule

$$\frac{\text{<Premises>}}{\text{<Context>} \vdash \text{<Expr>} : \text{<Type>}}$$

```
Expr ::= Var
       | lambda Var. Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
       | Expr if Expr else Expr
       | Expr BinaryOp Expr
       | true | false
       | 0 | 1 | 2 | …
```

```
Type ::= Int
       | Bool
       | List[Type]
       | Type -> Type
```

[Int] $$\frac{}{C \vdash 1 : \text{Int}}$$

[True] $$\frac{}{C \vdash \text{true} : \text{Bool}}$$

[False] $$\frac{}{C \vdash \text{false} : \text{Bool}}$$

Typing Rule

$$\frac{\text{<Premises>}}{\text{<Context>} \vdash \text{<Expr>} : \text{<Type>}}$$

```
Expr ::= Var
       | lambda Var. Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
       | Expr if Expr else Expr
       | Expr BinaryOp Expr
       | true | false
       | 0 | 1 | 2 | …
```

```
Type ::= Int
       | Bool
       | List[Type]
       | Type -> Type
```

[Arith] $$\frac{C \vdash \mathrm{e}_1 : \mathrm{Int} \quad C \vdash \mathrm{e}_2 : \mathrm{Int}}{C \vdash \mathrm{e}_1 + e_2 : \mathrm{Int}}$$

[Equality] $$\frac{C \vdash \mathrm{e}_1 : \mathrm{Int} \quad C \vdash \mathrm{e}_2 : \mathrm{Int}}{C \vdash \mathrm{e}_1 == e_2 : \mathrm{Bool}}$$

Typing Rule

$$\frac{\text{<Premises>}}{\text{<Context>} \vdash \text{<Expr>} : \text{<Type>}}$$

```
Expr ::= Var
       | lambda Var. Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
       | Expr if Expr else Expr
       | Expr BinaryOp Expr
       | true | false
       | 0 | 1 | 2 | …
```

```
Type ::= Int
       | Bool
       | List[Type]
       | Type -> Type
```

$$\text{[Var]}\ \frac{(v \mapsto \tau) \in C}{C \vdash \text{Var}(v) : \tau}$$

$$\text{[Lambda]}\ \frac{C \cup \{x \mapsto \tau_1\} \vdash e : \tau_2}{C \vdash \lambda x.e : \tau_1 \to \tau_2}$$

Typing Rule

$$\frac{\text{<Premises>}}{\text{<Context> } \vdash \text{ <Expr> : <Type>}}$$

```
Expr ::= Var
       | lambda Var. Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
       | Expr if Expr else Expr
       | Expr BinaryOp Expr
       | true | false
       | 0 | 1 | 2 | …
```

```
Type ::= Int
       | Bool
       | List[Type]
       | Type -> Type
```

[Map] $$\frac{C \vdash e_1 : \text{List}[\tau_1] \quad C \vdash e_2 : \tau_1 \to \tau_2}{C \vdash \text{map}(e_1, e_2) : \text{List}[\tau_2]}$$

[Filter] $$\frac{C \vdash e_1 : \text{List}[\tau_1] \quad C \vdash e_2 : \tau_1 \to \text{Bool}}{C \vdash \text{filter}(e_1, e_2) : \text{List}[\tau_1]}$$

Type Checking that Succeeds

```
len : [Int] -> Int
  len(list) = reduce(list, 0, lambda acc, n: acc + 1)
```

Type Checking that Succeeds

```
len : [Int] -> Int
  len(list) = reduce(list, 0, lambda acc, n: acc + 1)
```

$$\dfrac{\dots \quad \dfrac{(\text{list} \mapsto \text{List[Int]}) \in C}{C \vdash \text{list} : \text{List[Int]}} \quad \dfrac{}{C \vdash 0 : \text{Int}} \quad \dfrac{\dfrac{(\text{acc} \mapsto \text{Int}) \in C}{C \vdash \text{acc} : \text{Int}} \quad \dfrac{(\text{n} \mapsto \text{Int}) \in C}{C \vdash \text{n} : \text{Int}} \quad \dfrac{\dfrac{(\text{acc} \mapsto \text{Int}) \in C}{C \vdash \text{acc} : \text{Int}} \quad \dfrac{}{C \vdash 1 : \text{Int}}}{C \vdash \text{acc} + 1 : \text{Int}}}{C \vdash \text{lambda acc, n. acc} + 1 : \text{Int} \to \text{Int} \to \text{Int}}}{C \vdash \text{reduce(list, 0, lambda acc, n. acc} + 1) : \text{Int}}$$

$$C \vdash \text{len} : [\text{Int}] \to \text{Int}$$ ✅

Type Checking that Failed

```
gt0 : [Int] -> [Int]
  gt0(list) = filter(list, 0)
```

$$\text{[Filter]}\ \frac{C \vdash e_1 : \text{List}[\tau_1] \quad C \vdash e_2 : \tau_1 \to \text{Bool}}{C \vdash \text{filter}(e_1, e_2) : \text{List}[\tau_1]}$$

Type Checking that Failed

```
gt0 : [Int] -> [Int]
  gt0(list) = filter(list, 0)
```

[Filter] $\dfrac{C \vdash e_1 : \text{List}[\tau_1] \quad C \vdash e_2 : \tau_1 \to \text{Bool}}{C \vdash \text{filter}(e_1, e_2) : \text{List}[\tau_1]}$

$$\dfrac{\dfrac{(\text{list} \mapsto \text{List}[\text{Int}]) \in C}{C \vdash \text{list} : \text{List}[\text{Int}]} \qquad \dfrac{}{C \vdash 0 : \text{Int}}}{C \vdash \text{filter}(\text{list}, 0) : \text{❌}}$$

Type Checking that Succeeded (Corrected)

```
gt0 : [Int] -> [Int]
  gt0(list) = filter(list, lambda x: x > 0)
```

$$\text{[Filter]}\ \frac{C \vdash e_1 : \text{List}[\tau_1] \quad C \vdash e_2 : \tau_1 \to \text{Bool}}{C \vdash \text{filter}(e_1, e_2) : \text{List}[\tau_1]}$$

$$\dfrac{\dfrac{(\text{list} \mapsto \text{List[Int]}) \in C}{C \vdash \text{list} : \text{List[Int]}} \quad \dfrac{\dfrac{\dfrac{(\text{x} \mapsto \text{Int}) \in C}{C \vdash \text{x} : \text{Int}} \quad \dfrac{}{C \vdash 0 : \text{Int}}}{C \vdash x > 0 : \text{Bool}} \quad \dfrac{(\text{x} \mapsto \text{Int}) \in C}{C \vdash \text{x} : \text{Int}}}{C \vdash \text{lambda } x.\, x > 0 : \text{Int} \to \text{Bool}}}{\dfrac{\dots \qquad C \vdash \text{filter(list, lambda x. x} > 0) : \text{List[Int]}}{C \vdash \text{gt0} : \text{List[Int]} \to \text{List[Int]}}}$$

✅

Typing Rules

```
Expr ::= Var
       | lambda Var. Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
       | Expr if Expr else Expr
       | Expr BinaryOp Expr
       | true | false
       | 0 | 1 | 2 | …
Type ::= Int
       | Bool
       | List[Type]
       | Type -> Type
```

$$\text{[Int]}\ \frac{}{C \vdash \text{IntLiteral} : \text{Int}}$$

$$\text{[True]}\ \frac{}{C \vdash \text{true} : \text{Bool}}$$

$$\text{[False]}\ \frac{}{C \vdash \text{false} : \text{Bool}}$$

$$\text{[Arith]}\ \frac{C \vdash e_1 : \text{Int} \quad C \vdash e_2 : \text{Int}}{C \vdash e_1 + e_2 : \text{Bool}}$$

$$\text{[Equality]}\ \frac{C \vdash e_1 : \text{Int} \quad C \vdash e_2 : \text{Int}}{C \vdash e_1 == e_2 : \text{Bool}}$$

$$\text{[Var]}\ \frac{(v \mapsto \tau) \in C}{C \vdash \text{Var}(v) : \tau}$$

$$\text{[Lambda]}\ \frac{C \cup \{x \mapsto \tau_1\} \vdash e : \tau_2}{C \vdash \lambda x.\, e : \tau_1 \to \tau_2}$$

$$\text{[Map]}\ \frac{C \vdash e_1 : \text{List}[\tau_1] \quad C \vdash e_2 : \tau_1 \to t_2}{C \vdash \text{map}(e_1, e_2) : \text{List}[\tau_2]}$$

$$\text{[Filter]}\ \frac{C \vdash e_1 : \text{List}[\tau_1] \quad C \vdash e_2 : \tau_1 \to \text{Bool}}{C \vdash \text{filter}(e_1, e_2) : \text{List}[\tau_1]}$$

Typing Rules in the Wild

Datafun [Arntzenius et. al. 2016]

$$\frac{\Delta;\Gamma \vdash \mathcal{L} \Rightarrow \Delta' \quad \Delta,\Delta';\Gamma \vdash e : \mathrm{L}}{\Delta;\Gamma \vdash \bigvee(\mathcal{L})\ e : \mathrm{L}}$$

$$\frac{\Delta;\Gamma \vdash \mathcal{L} \Rightarrow \Delta' \quad \Delta,\Delta';\Gamma \vdash e : \mathrm{A}}{\Delta;\Gamma \vdash \{e \mid \mathcal{L}\} : \{\mathrm{A}\}}$$

$$\frac{\Delta;\cdot \vdash e : \mathrm{A} \quad \Delta;\Gamma \vdash p : \mathrm{A} \Rightarrow \Delta' \quad \Delta,\Delta';\Gamma \vdash e_1 : \mathrm{C} \quad \Delta;\Gamma}{\Delta;\Gamma \vdash \mathsf{case}\ e\ \mathsf{of}\ p \to e_1;\ \_ \to e_2 : \mathrm{C}}$$

$$\frac{\Delta;\Gamma \vdash \mathcal{L}_1 \Rightarrow \Delta_1 \quad \Delta;\Gamma \vdash \mathcal{L}_2 \Rightarrow}{\Delta;\Gamma \vdash \mathcal{L}_1,\mathcal{L}_2 \Rightarrow \Delta_1,\Delta_2}$$

$$\frac{\Delta;\Gamma \vdash e : \{\mathrm{A}\} \quad \Delta;\Gamma \vdash p : \mathrm{A} \Rightarrow \Delta}{\Delta;\Gamma \vdash p \in e \Rightarrow \Delta'}$$

$$\frac{}{\Delta;\Gamma \vdash \_ : \mathrm{A}}$$

$$\frac{\Delta;\Gamma \vdash p_1 : \mathrm{A}_1 \Rightarrow \Delta_1 \quad \Delta;}{\Delta;\Gamma \vdash (p_1,p_2) : \mathrm{A}_1 \times}$$

$$\frac{\Delta;\Gamma \vdash e : \underset{eq}{\mathrm{A}}}{\Delta;\Gamma \vdash !e : \underset{eq}{\mathrm{A}} \Rightarrow \cdot} \qquad \frac{}{\Delta;\Gamma \vdash}$$

WebAssembly [Haas et. al. 2017]

(contexts) $C ::= \{\mathsf{func}\ tf^*,\ \mathsf{global}\ tg^*,\ \mathsf{table}\ n^?,\ \mathsf{memory}\ n^?,\ \mathsf{local}\ t^*,\ \mathsf{label}\ (t^*)^*,\ \mathsf{return}\ (t^*)^?\}$

Typing Instructions $\boxed{C \vdash e^* : tf}$

$$\frac{}{C \vdash t.\mathbf{const}\ c : \epsilon \to t} \qquad \frac{}{C \vdash t.unop : t \to t} \qquad \frac{}{C \vdash t.binop : t\ t \to t} \qquad \frac{}{C \vdash t.testop : t \to \mathsf{i32}} \qquad \frac{}{C \vdash t.relop : t\ t \to \mathsf{i32}}$$

$$\frac{t_1 \neq t_2 \qquad sx^? = \epsilon \Leftrightarrow (t_1 = \mathbf{i}n \wedge t_2 = \mathbf{i}n' \wedge |t_1| < |t_2|) \vee (t_1 = \mathbf{f}n \wedge t_2 = \mathbf{f}n')}{C \vdash t_1.\mathbf{convert}\ t_2\_sx^? : t_2 \to t_1} \qquad \frac{t_1 \neq t_2 \qquad |t_1| = |t_2|}{C \vdash t_1.\mathbf{reinterpret}\ t_2 : t_2 \to t_1}$$

$$\frac{}{C \vdash \mathbf{drop} : t \to \epsilon} \qquad \frac{}{C \vdash \mathbf{select} : t\ t\ \mathsf{i32} \to t}$$

$$\frac{tf = t_1^n \to t_2^m \qquad C, \mathsf{label}\ (t_1^n) \vdash e^* : tf}{C \vdash \mathbf{loop}\ tf\ e^*\ \mathbf{end} : tf}$$

$$\frac{\vdash e_1^* : tf \qquad C, \mathsf{label}\ (t_2^m) \vdash e_2^* : tf}{e_2^*\ \mathbf{end} : t_1^n\ \mathsf{i32} \to t_2^m}$$

$$\frac{t^*}{32 \to t^*} \qquad \frac{(C_{\mathsf{label}}(i) = t^*)^+}{C \vdash \mathbf{br\_table}\ i^+ : t_1^*\ t^*\ \mathsf{i32} \to t_2^*}$$

$$\frac{= tf}{: tf} \qquad \frac{tf = t_1^* \to t_2^* \qquad C_{\mathsf{table}} = n}{C \vdash \mathbf{call\_indirect}\ tf : t_1^*\ \mathsf{i32} \to t_2^*}$$

$\Sigma;\Delta;\Gamma;\Theta \vdash e : \tau \Rightarrow \Gamma'$

T-U32

$$\frac{}{\Sigma;\Delta;\Gamma;\Theta \vdash \boxed{n} : \mathsf{u32} \Rightarrow \Gamma}$$

T-Move

$$\frac{\Delta;\Gamma;\Theta \vdash_{\mathsf{uniq}} \pi \Rightarrow \{\ ^{\mathsf{uniq}}\pi\ \} \quad \Gamma(\pi) = \tau^{\mathrm{SI}} \quad \mathbf{noncopyable}_\Sigma\ \tau^{\mathrm{SI}}}{\Sigma;\Delta;\Gamma;\Theta \vdash \boxed{\pi} : \tau^{\mathrm{SI}} \Rightarrow \Gamma[\pi \mapsto \tau^{\mathrm{SI}^\dagger}]}$$

T-Copy

$$\frac{\Delta;\Gamma;\Theta \vdash_{\mathsf{shrd}} p \Rightarrow \{\bar{\ell}\} \quad \Delta;\Gamma \vdash_{\mathsf{shrd}} p : \tau^{\mathrm{SI}} \quad \mathbf{copyable}_\Sigma\ \tau^{\mathrm{SI}}}{\Sigma;\Delta;\Gamma;\Theta \vdash \boxed{p} : \tau^{\mathrm{SI}} \Rightarrow \Gamma}$$

T-Borrow

$$\frac{\Gamma(r) = \emptyset \qquad \Gamma;\Theta \vdash r\ \mathsf{rnic} \quad \Delta;\Gamma;\Theta \vdash_\omega p \Rightarrow \{\bar{\ell}\} \qquad \Delta;\Gamma \vdash_\omega p : \tau^{\mathrm{XI}}}{\Sigma;\Delta;\Gamma;\Theta \vdash \boxed{\&r\ \omega\ p} : \&r\ \omega\ \tau^{\mathrm{XI}} \Rightarrow \Gamma[r \mapsto \{\bar{\ell}\}]}$$

T-LetRegion

$$\frac{\Sigma;\Delta;\Gamma,\ r \mapsto \{\};\Theta \vdash \boxed{e} : \tau^{\mathrm{SI}} \Rightarrow \Gamma',\ r \mapsto \{\bar{\ell}\}}{\Sigma;\Delta;\Gamma;\Theta \vdash \boxed{\mathtt{letrgn}\ \langle r \rangle\ \{\ e\ \}} : \tau^{\mathrm{SI}} \Rightarrow \Gamma'}$$

T-Branch

$$\frac{\Sigma;\Delta;\Gamma;\Theta \vdash \boxed{e_1} : \mathtt{bool} \Rightarrow \Gamma_1 \qquad \Sigma;\Delta;\Gamma_1;\Theta \vdash \boxed{e_2} : \tau_2^{\mathrm{SI}} \Rightarrow \Gamma_2 \quad \Sigma;\Delta;\Gamma_1;\Theta \vdash \boxed{e_3} : \tau_3^{\mathrm{SI}} \Rightarrow \Gamma_3 \qquad \tau^{\mathrm{SI}} = \tau_2^{\mathrm{SI}} \vee \tau^{\mathrm{SI}} = \tau_3^{\mathrm{SI}} \quad \Delta;\Gamma_2;\Theta \vdash^+ \tau_2^{\mathrm{SI}} \rightsquigarrow \tau^{\mathrm{SI}} \dashv \Gamma_2' \qquad \Delta;\Gamma_3;\Theta \vdash^+ \tau_3^{\mathrm{SI}} \rightsquigarrow \tau^{\mathrm{SI}} \dashv \Gamma_3' \qquad \Gamma_2' \uplus \Gamma_3' = \Gamma'}{\Sigma;\Delta;\Gamma;\Theta \vdash \boxed{\mathtt{if}\ e_1\ \{\ e_2\ \}\ \mathtt{else}\ \{\ e_3\ \}} : \tau^{\mathrm{SI}} \Rightarrow \Gamma'}$$

T-Seq

$$\frac{\Sigma;\Delta;\Gamma;\Theta \vdash \boxed{e_1} : \tau_1^{\mathrm{SI}} \Rightarrow \Gamma_1 \quad \Sigma;\Delta;\mathrm{gc\text{-}loans}_\Theta(\Gamma_1);\Theta \vdash \boxed{e_2} : \tau_2^{\mathrm{SI}} \Rightarrow \Gamma_2}{\Sigma;\Delta;\Gamma;\Theta \vdash \boxed{e_1;\ e_2} : \tau_2^{\mathrm{SI}} \Rightarrow \Gamma_2}$$

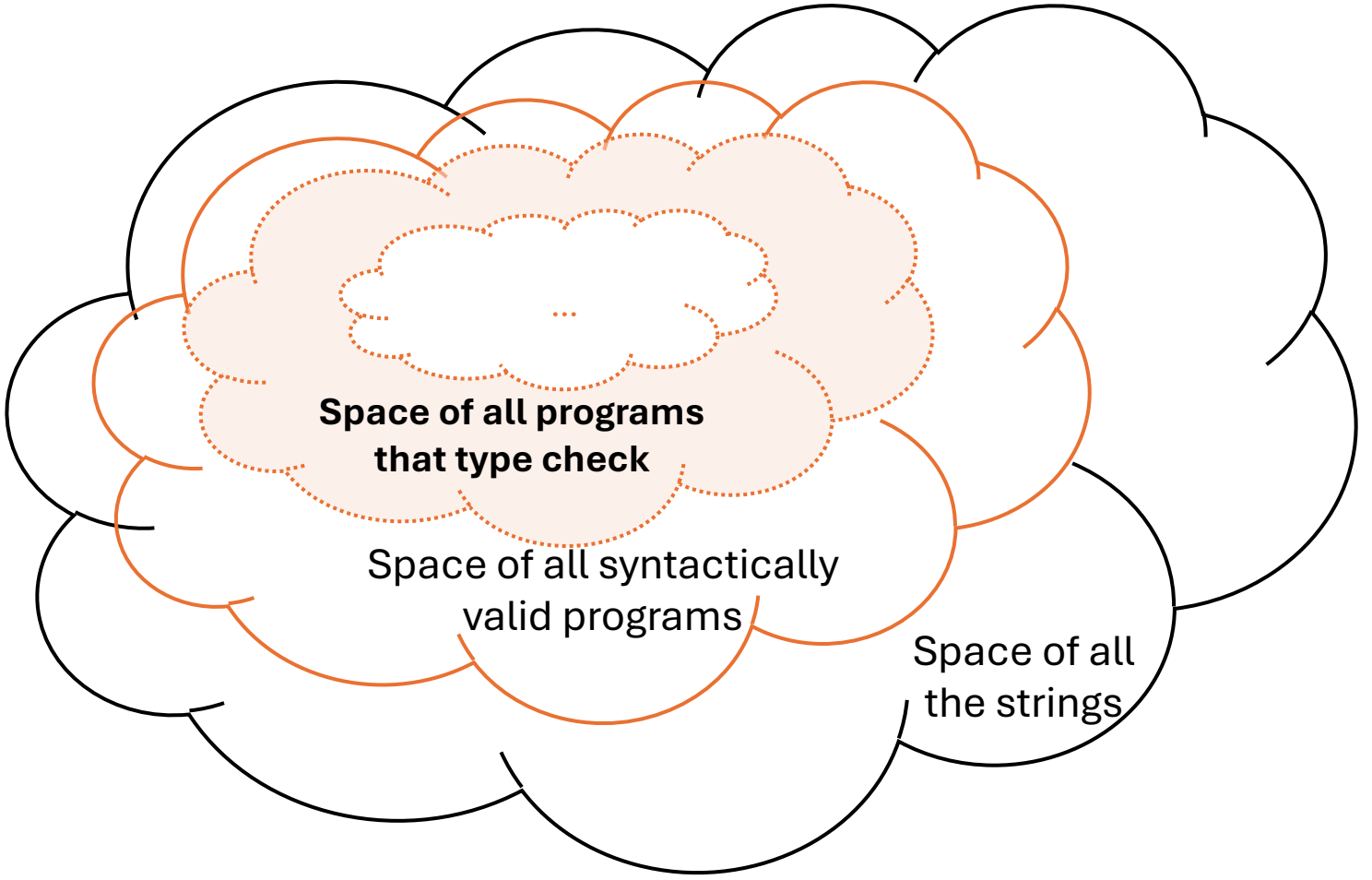
T-Let

$$\frac{\Sigma;\Delta;\Gamma;\Theta \vdash \boxed{e_1} : \tau_1^{\mathrm{SI}} \Rightarrow \Gamma_1 \qquad \Delta;\Gamma_1;\Theta \vdash^+ \tau_1^{\mathrm{SI}} \rightsquigarrow \tau_a^{\mathrm{SI}} \dashv \Gamma_1' \quad \forall r \in \text{free-regions}(\tau_a^{\mathrm{SI}}).\ \Gamma_1' \vdash r\ \mathtt{rnrb} \qquad \Sigma;\Delta;\mathrm{gc\text{-}loans}_\Theta(\Gamma_1',\ x\ :\ \tau_a^{\mathrm{SI}});\Theta \vdash \boxed{e_2} : \tau_2^{\mathrm{SI}} \Rightarrow \Gamma_2,\ x\ :\ \tau^{\mathrm{SD}}}{\Sigma;\Delta;\Gamma;\Theta \vdash \boxed{\mathtt{let}\ x : \tau_a^{\mathrm{SI}} = e_1;\ e_2} : \tau_2^{\mathrm{SI}} \Rightarrow \Gamma_2}$$

Rust (Oxide) [Weiss et. al. 2021]

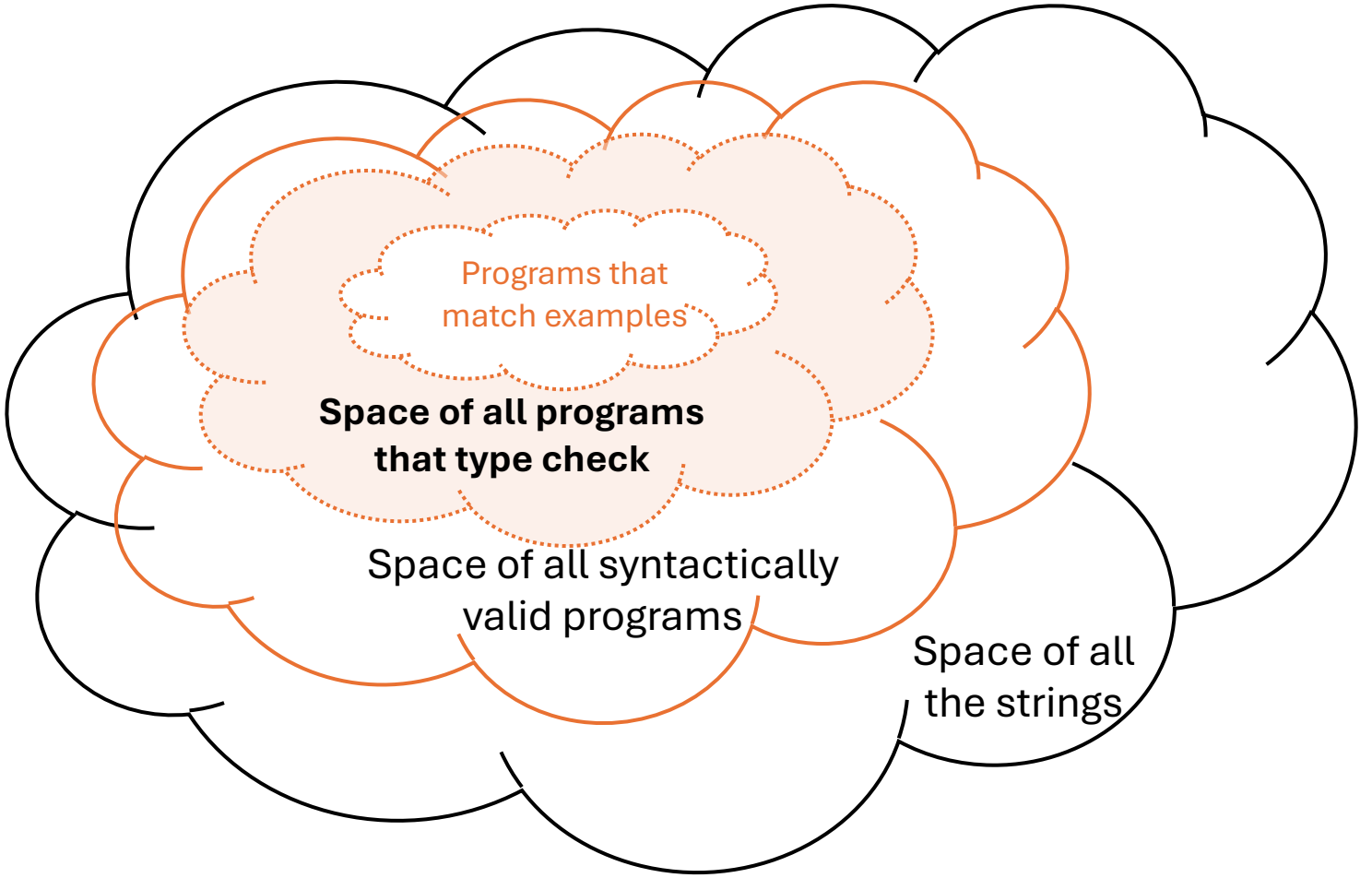
Typing Checking

```
Expr ::= Var
       | lambda Var: Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
       | Expr if Expr else Expr
       | Expr BinaryOp Expr
       | true | false
       | 0 | 1 | 2 | …
```



Typing Checking

```
Expr ::= Var
       | lambda Var: Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
       | Expr if Expr else Expr
       | Expr BinaryOp Expr
       | true | false
       | 0 | 1 | 2 | …
```



Top-Down Enumerative Search

```
Expr ::= Var
       | lambda Var: Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
       | Expr if Expr else Expr
       | Expr BinaryOp Expr
       | true | false
       | 0 | 1 | 2 | …
```

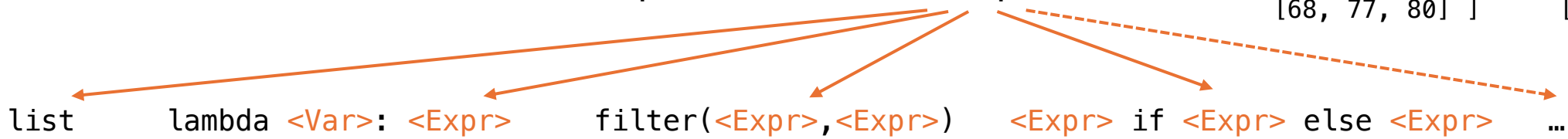
```
dropmin(list) = <Expr>
```

```
[ [71, 75, 83],     [ [75, 83],
  [90, 87, 95], →     [90, 95],
  [68, 77, 80] ]      [77, 80] ]
```

Top-Down Enumerative Search

```
Expr ::= Var
       | lambda Var: Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
       | Expr if Expr else Expr
       | Expr BinaryOp Expr
       | true | false
       | 0 | 1 | 2 | …
```

```
[ [71, 75, 83],      [ [75, 83],
  [90, 87, 95], →      [90, 95],
  [68, 77, 80] ]       [77, 80] ]
```



Top-Down Enumerative Search

```
Expr ::= Var
       | lambda Var: Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
       | Expr if Expr else Expr
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       | true | false
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```

```
[ [71, 75, 83],      [ [75, 83],
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```

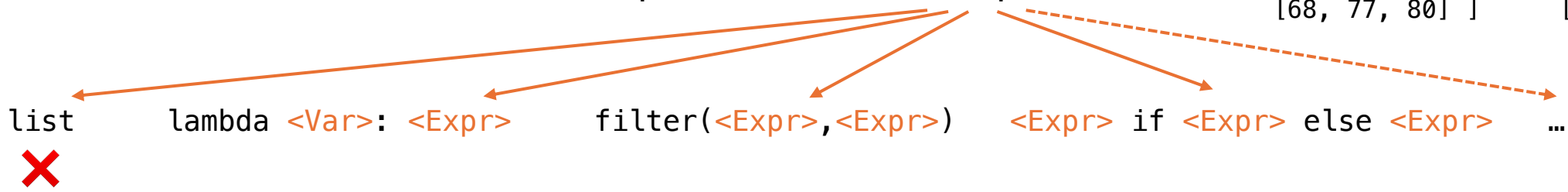

Top-Down Enumerative Search

```
Expr ::= Var
       | lambda Var: Expr
       | filter(Expr, Expr)
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```
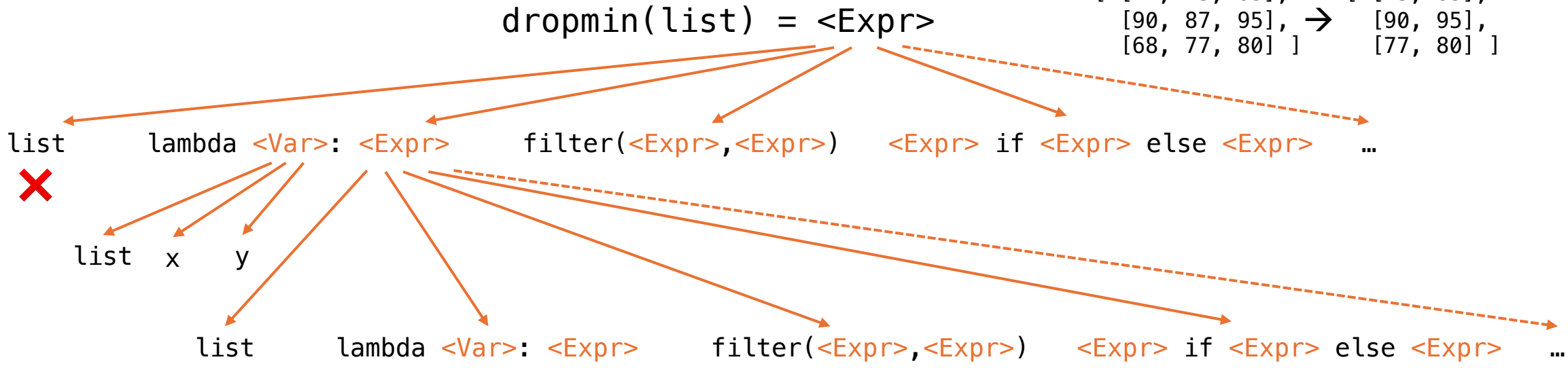

```
[ [71, 75, 83],     [ [75, 83],
  [90, 87, 95], →     [90, 95],
  [68, 77, 80] ]      [77, 80] ]
```



Top-Down Enumerative Search

```
Expr ::= Var
       | lambda Var: Expr
       | filter(Expr, Expr)
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```

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[ [71, 75, 83],      [ [75, 83],
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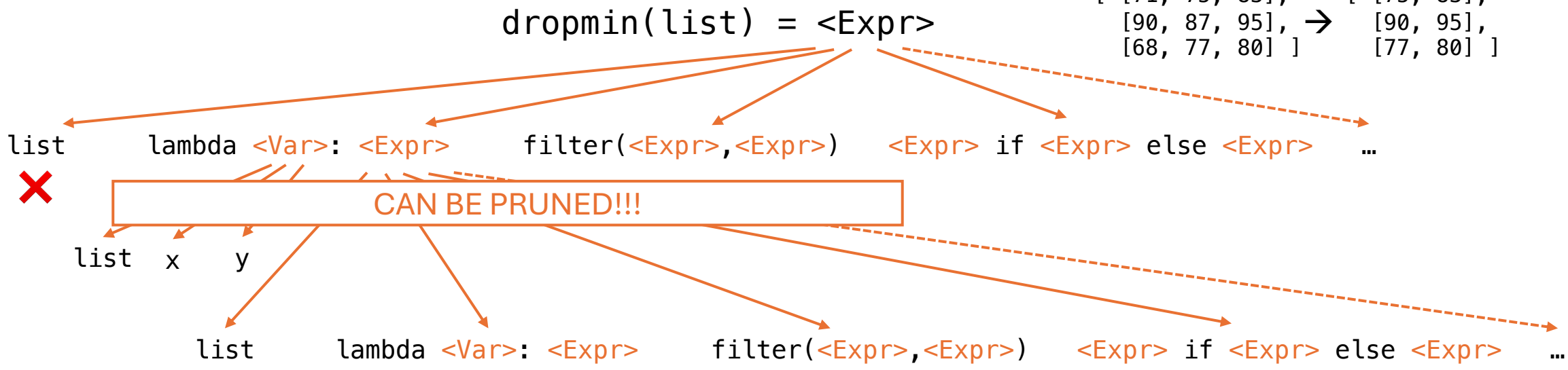
```



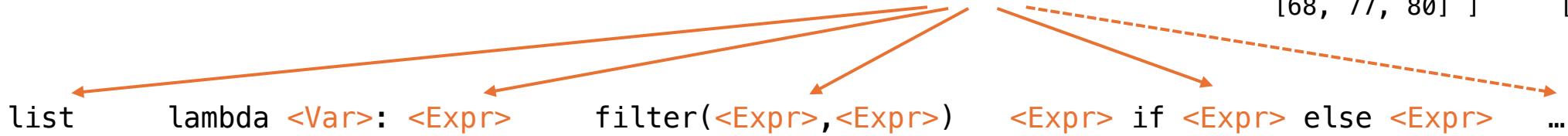
Type-Guided Top-Down Search

```
Expr ::= Var
       | lambda Var: Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
       | Expr if Expr else Expr
       | Expr BinaryOp Expr
       | true | false
       | 0 | 1 | 2 | …
```

dropmin : List[List[Int]] -> List[List[Int]]

```
[ [71, 75, 83],      [ [75, 83],
  [90, 87, 95], →      [90, 95],
  [68, 77, 80] ]       [77, 80] ]
```

dropmin(list) = <Expr>



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```

dropmin : List[List[Int]] -> List[List[Int]]

dropmin(list) = <Expr>

```
[ [71, 75, 83],      [ [75, 83],
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```

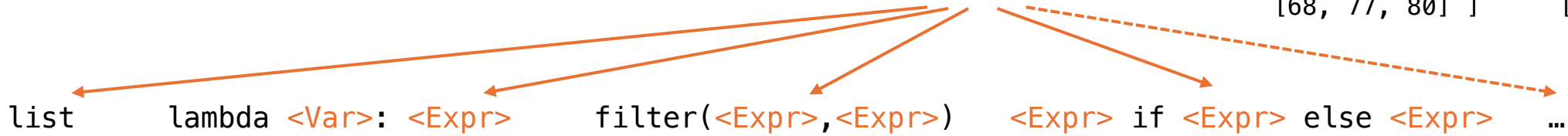

Type-Guided Top-Down Search

```
Expr ::= Var
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dropmin : List[List[Int]] -> List[List[Int]]

```
[ [71, 75, 83],      [ [75, 83],
  [90, 87, 95], →      [90, 95],
  [68, 77, 80] ]       [77, 80] ]
```

dropmin(list) = <Expr>

list lambda <Var>: <Expr> filter(<Expr>,<Expr>) <Expr> if <Expr> else <Expr> …

List[List[Int]]

$$\frac{(\text{list} \mapsto \text{List[Int]}) \in C}{C \vdash \text{list} : \text{List[Int]}}$$

Type-Guided Top-Down Search

```
Expr ::= Var
       | lambda Var: Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
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```

```
dropmin : List[List[Int]] -> List[List[Int]]
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[ [71, 75, 83],    [ [75, 83],
  [90, 87, 95], →    [90, 95],
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dropmin(list) = <Expr>
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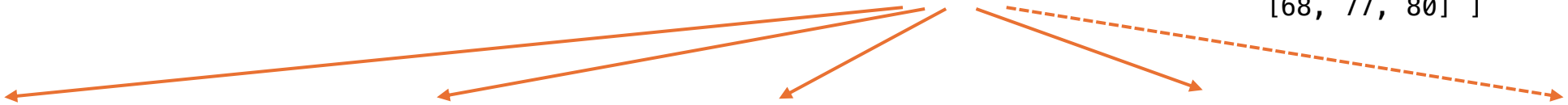

List[List[Int]] ✅

```
[ [71, 75, 83],    [ [75, 83],
  [90, 87, 95], →    [90, 95],
  [68, 77, 80] ]     [77, 80] ]
```

❌

Type-Guided Top-Down Search

```
Expr ::= Var
       | lambda Var: Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
       | Expr if Expr else Expr
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       | true | false
       | 0 | 1 | 2 | …
```

dropmin : List[List[Int]] -> List[List[Int]]

```
dropmin(list) = <Expr>
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[ [71, 75, 83],      [ [75, 83],
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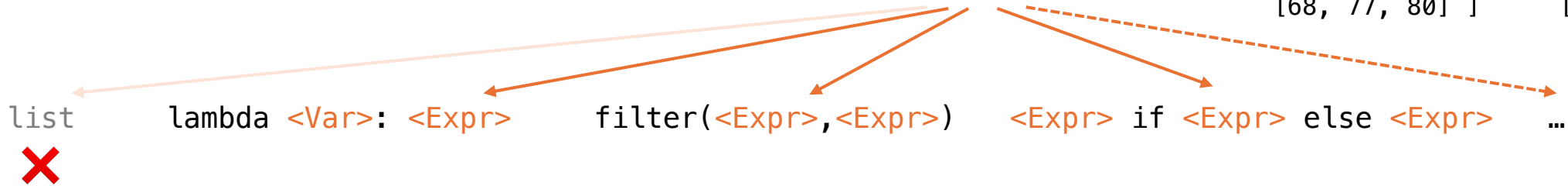

Type-Guided Top-Down Search

```
Expr ::= Var
       | lambda Var: Expr
       | filter(Expr, Expr)
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dropmin(list) = <Expr>
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[ [71, 75, 83],      [ [75, 83],
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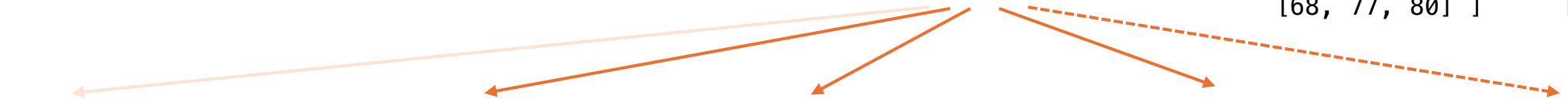

❌ **List[List[Int]]**

[Lambda] $\dfrac{C \cup \{x \mapsto \tau_1\} \vdash e : \tau_2}{C \vdash \lambda x.e : \tau_1 \to \tau_2}$

Type-Guided Top-Down Search

```
Expr ::= Var
       | lambda Var: Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
       | Expr if Expr else Expr
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dropmin(list) = <Expr>
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```

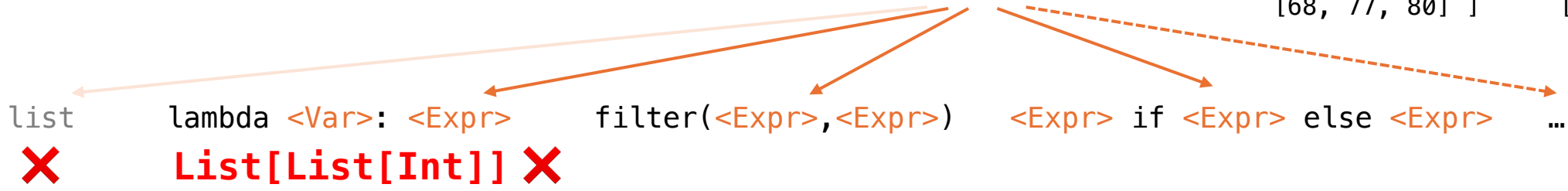

Type-Guided Top-Down Search

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Expr ::= Var
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       | filter(Expr, Expr)
       | map(Expr, Expr)
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       | Expr if Expr else Expr
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       | 0 | 1 | 2 | …
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```
[ [71, 75, 83],      [ [75, 83],
  [90, 87, 95], →      [90, 95],
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```
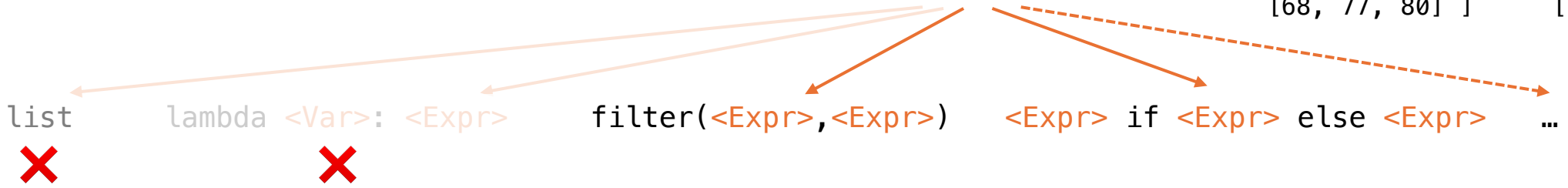

dropmin : List[Int] -> List[Int]

dropmin(list) = <Expr>



Type-Guided Top-Down Search

```
Expr ::= Var
       | lambda Var: Expr
       | filter(Expr, Expr)
       | map(Expr, Expr)
       | reduce(Expr, Expr, Expr)
       | Expr if Expr else Expr
       | Expr BinaryOp Expr
       | true | false
       | 0 | 1 | 2 | …
```

```
[ [71, 75, 83],      [ [75, 83],
  [90, 87, 95], →      [90, 95],
  [68, 77, 80] ]       [77, 80] ]
```

dropmin : List[Int] -> List[Int]

dropmin(list) = <Expr>

Polymorphism

Base types

```
Int
```

Parametric types (Polymorphism)

```
∀T, List[T]
```

Parametric type instance (Concrete; Monomorphic)

```
List[Int]
```

Polymorphism

Base types

```
Int
```

Parametric types (Polymorphism)

```
∀T, List[T]
```

Parametric type instance (Concrete; Monomorphic)

```
List[Int]
```

Types as sets

Base types

```
Int      = {…, -2, -1, 0, 1, 2, …}
```

Parametric types (Polymorphism)

```
∀T, List[T]      = {…,[], [“a”, “b”], [3], [false, false, true], …}
```

Parametric type instance (Concrete; Monomorphic)

```
List[Int]      = {[], [1], [2, 3], [1, 2, 3], …}
```

Types as sets that can be refined!

Base types

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Int      = {…, -2, -1, 0, 1, 2, …}
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Parametric types (Polymorphism)

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∀T, List[T]   = {…,[], [“a”, “b”], [3], [false, false, true], …}
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Parametric type instance (Concrete; Monomorphic)

```
List[Int]     = {[], [1], [2, 3], [1, 2, 3], …}
```

Refinement types

```
{v : Int | v > 0}
```

Types as sets that can be refined!

Base types

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Int      = {…, -2, -1, 0, 1, 2, …}
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Parametric types (Polymorphism)

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∀T, List[T]   = {…,[], [“a”, “b”], [3], [false, false, true], …}
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Refinement types

```
{v : Int | v > 0}   = {1, 2, 3, …}
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Types as sets that can be refined!

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Int      = {…, -2, -1, 0, 1, 2, …}
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Parametric types (Polymorphism)

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∀T, List[T]   = {…,[], [“a”, “b”], [3], [false, false, true], …}
```

Parametric type instance (Concrete; Monomorphic)

```
List[Int]     = {[], [1], [2, 3], [1, 2, 3], …}
```

Refinement types

```
{v : Int | v > 0}   = {1, 2, 3, …}
∀T, {ls : List[T] | len(ls) > 0}  = {[1], [2, 3], [1, 2, 3], …}
```

Refinement Types for Synthesis

“absolute value”

Typing Specification (Original)

```
abs :: Int -> Int
```

Typing Specification (with Refinement)

```
abs :: Int -> {v : Int | v >= 0}
```

Refinement Types for Synthesis

“duplicate every element in a list”

Typing Specification (Original)

```
stutter :: List[T] -> List[T]
```

Typing Specification (with Refinement)

```
stutter :: in: List[T] -> {v: List[T] | len(v) = 2 * len(x)}
```

Refinement Types for Synthesis

“sort the elements in an array”

Typing Specification (Original)

```
sort :: List[T] -> List[T]
```

Typing Specification (with Refinement)

```
sort :: List[T] -> {v: List[T] | is_sorted(v)}
```

Refinement Types for Synthesis

“sort the elements in an array”

Typing Specification (Original)

```
sort :: List[T] -> List[T]
```

Typing Specification (with Refinement)

```
sort :: List[T] -> {v: List[T] | is_sorted(v)}
is_sorted(v) = ∀i,j, s.t., 0 <= i < j < len(v), v[i] <= v[j]
```

Refinement Types for Synthesis

“insert an element into an array”

Typing Specification (Original)

```
insert :: T -> List[T] -> List[T]
```

Typing Specification (with Refinement)

```
insert :: (e: T) -> (x: List[T]) -> {y : List[T] | elems(y) = elems(x) U {e}}
```

How to Enforce Constraints in Refinement Types

- similar strategies in top-down enumerative synthesis
- with type checkers capable of checking and strengthening refinement types
- during the type checking, we may rely on human defined predicates such as “len” and “elems”

How to Enforce Constraints in Refinement Types

Leveraging Rust Types for Program Synthesis

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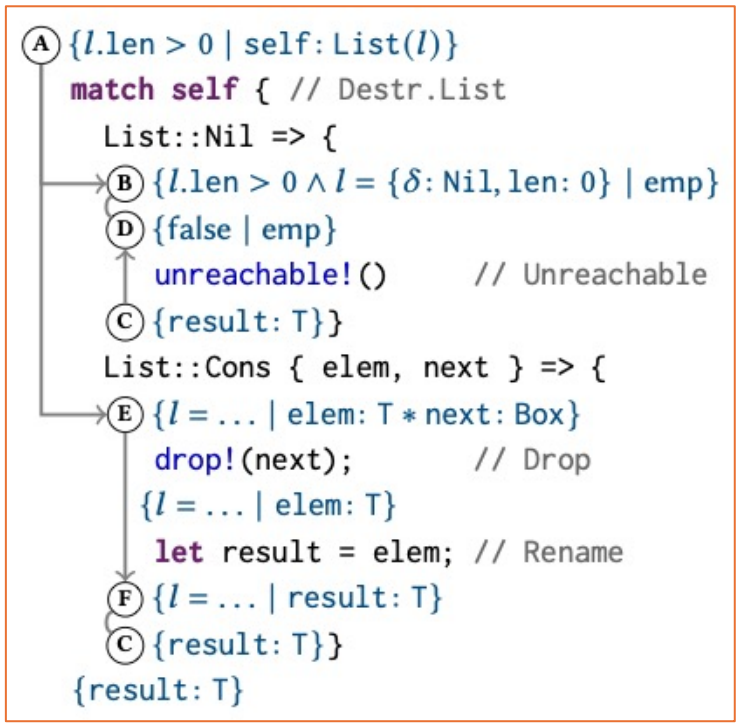
ILYA SERGEY, National University of Singapore, Singapore

```
#[ensures((^self).len() ==
          (*self).len() + (^result).len())]
fn last_mut(&mut self) -> &mut List<T> {
  todo!()
}
```



Summary

- Functional Language with Higher-Order Functions
- Type System, Typing Rules, and Type Checking
- Top-down Enumerative Synthesis Guided by Types
- (A little bit of) Refinement Types for Synthesis

Week 2

- Assignment 1
  - Released: https://github.com/machine-programming/assignment-1
  - Due Thursday of the Third Week (Sep 11)
  - Autograder not up yet; will be later today
  - API keys were sent out
- Waitlisted students
  - Please contact me by sending emails; will add you to Courselore, GradeScope, and give you API keys
- Any questions?